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ON THE MAXIMAL DISTANCE BETWEEN TWO RENEWAL EPOCHS

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Let $X_1, X_2 \dots$ be a sequence of positive, independent, identically distributed (i.i.d.) random variables with $S_0=0$, $S_n=X_1+\cdots+X_n$, $n\geq 1$. Denote by $\tau_t=\sup\{n \mid S_n\leq t\}$. In this paper we establish almost sure lower and upper bounds for $M_t=\max\{X_1, X_2, \dots, X_{\tau_t}, t-S_{\tau_t}\}$ if the underlying distribution function has a regularly varying tail.

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renewal processes * almost sure limit laws

1. Introduction

Let $X_1, X_2, \dots$ be a sequence of positive i.i.d. random variables with $P(X_1\leq x)=F(x)$. Further let $S_0=0$, $S_n=X_1+\cdots+X_n$, $n\geq 1$ and let τ_t be the largest integer for which $S_{\tau_t}\leq t$. The main goal of this paper is to study the properties of the process

$$M_t=M(t)=\max\{X_1, X_2, \dots, X_{\tau_t}, t-S_{\tau_t}\}.$$

M_t can be interpreted in its proper context, for example in a queueing process, it might represent the longest interval up to t without arrival of a customer.

The present problem is originated by a problem in random walk theory: in fact, let $R_1, R_2, \dots$ be a sequence of i.i.d. random variables with $P(R_1=1)=\frac{1}{2}=P(R_1=-1)$ and put

$$Y_0=0, \qquad Y_n=R_1+R_2+\cdots+R_n \qquad (n\geq 1),$$

$$\rho_0=0, \qquad \rho_1=\min\{i>0;\ Y_i=0\},$$

$$\rho_{j+1}=\min\{i>\rho_j;\ Y_i=0\} \quad (j\geq 1).$$

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Let μ_t be the largest integer for which $\rho_{\mu_t} \leq t$. Then the length of the largest (not surely completed) excursion of the random walk $\{Y_n\}$ is given by

$$\varepsilon_t = \max\{\rho_1, \rho_2 - \rho_1, \ldots, \rho_{\mu_t} - \rho_{\mu_t - 1}, t - \rho_{\mu_t}\}.$$

It is clear that the problem of studying the properties of ε_t is a special case of the problem mentioned above. In fact, if $X_i = \rho_i - \rho_{i-1}$, then $M_t = \varepsilon_t$. We now summarize some of the results describing the behaviour of ε_t as $t \to \infty$. We start with a result of Chung and Erdös [4], giving the best possible upper bound for ε_t.

Theorem A [4; 5, p. 366]. *Let f be a non-decreasing positive function for which $\lim_{x\to\infty} f(x) = \infty$ and put*

$$I(f) = \int_1^\infty \frac{\mathrm{d}x}{x f^{1/2}(x)}.$$

Then

$$P\left\{\varepsilon_n \geq n\left(1 - \frac{1}{f(n)}\right) \text{ i.o.}\right\} = \begin{cases} 1 & \text{if } I(f) = \infty, \\ 0 & \text{if } I(f) < \infty. \end{cases} \tag{1.1}$$

The best possible lower bound for ε_n was established by Csáki et al. [5]. They showed that

$$\liminf_{n\to\infty} \varepsilon_n \frac{\log\log n}{n} = \beta \quad \text{a.s.} \tag{1.2}$$

where β is the root of the equation

$$\sum_{k=1}^\infty \frac{\beta^k}{k!(2k-1)} = 1.$$

(1.1) implies that

$$\varepsilon_n \geq n - \frac{n}{(\log n)^2}$$

infinitely often with probability one. This shows that for some n, the random walk $\{Y_n\}$ is essentially one excursion. On the other hand, (1.2) shows that for some n, the length of the longest excursion is only $\beta n(\log\log n)^{-1}$, which implies that in these cases the random walk consists of at least $\beta^{-1} \log\log n$ excursions. The main theorems of the present paper extend the results in (1.1) and (1.2) to M_t.

Throughout the paper we assume that the underlying distribution function (d.f.) F has a regularly varying tail, i.e.

$$1 - F(x) = \frac{x^{-\alpha}}{L(x)}, \quad x > 0, \ \alpha \geq 0, \tag{1.3}$$

with $L:\mathbb{R}^+\to\mathbb{R}^+$ slowly varying ($L\in SV$), which means that L is measurable, positive and

$$\lim_{x\to\infty}\frac{L(xt)}{L(x)}=1\quad\text{for every } t>0.$$

The number $-\alpha$ in (1.3) is called the index of regular variation of $1-F$.

Remark. In case of the random walk $\{Y_n\}$, we have that

$$P(\rho_1=2k)=\binom{2k-2}{k-1}\frac{1}{2^{2k-1}\cdot k}\approx\frac{k^{-3/2}}{2\sqrt{\pi}}\quad(k\to\infty)$$

which means that ε_n is just a special version of M_t when the underlying d.f. F has a regularly varying tail of index $-\frac{1}{2}$.

In the sequel we write

$$L_\alpha(x):=(L^{1/\alpha})^*(x^{1/\alpha}),\quad x>0,\ \alpha>0, \tag{1.4}$$

where L^* denotes the so called conjugate slowly varying function to L, see [3, 8]. With the definition in (1.4), it follows from de Bruyn [8] that

$$x^\alpha L(x)\sim y(x\to\infty)$$

if and only if

$$x\sim y^{1/\alpha}L_\alpha(y)\quad(y\to\infty),$$

To formulate our results, we also need a refined version of slow variation, provided by the so called π-variation (πV): $L:\mathbb{R}^+\to\mathbb{R}$ is π-varying ($L\in\pi V$) if and only if there exists a function $a\in SV$ such that

$$\lim_{x\to\infty}\frac{L(xt)-L(x)}{a(x)}=\log t,\quad t>0. \tag{1.5}$$

If (1.5) holds, de Haan [9] showed that it is always possible to take

$$a(x)=C(x):=L(x)-\frac{1}{x}\int_0^x L(y)\,\mathrm{d}y. \tag{1.6}$$

The concept of π-variation was in its turn refined by Omey and Willekens [15]. They studied functions L satisfying

$$\lim_{x\to\infty}\frac{L(xt)-L(x)-a(x)\log t}{b(x)}=\Lambda(t) \tag{1.7}$$

for some functions a, b and Λ.

If L satisfies (1.7), we call it π-varying with remainder ($L \in \pi R$). It was shown in [15] that (1.7) holds with $b \in SV$ if and only if

$$\lim_{x\to\infty} \frac{C(xt)-C(x)}{b(x)} = \log t, \quad t>0, \tag{1.8}$$

where C is the function defined in (1.6).

Furthermore if (1.7) holds, we can always take $a(x)=C(x)$ and $\Lambda(t)=\log t+\frac{1}{2}(\log t)^2$, see [15].

We are now ready to formulate our main results, presented in the next section. The proofs are collected in Section 3.

2. Main results

We distinguish between the two cases $\mu := \int_0^\infty x\,\mathrm{d}F(x) < \infty$ and $\mu=\infty$. First consider the case $\mu<\infty$. It then follows from the strong law of large numbers that

$$\frac{\tau_t}{t} \to \frac{1}{\mu} \quad \text{a.s.}$$

Hence M_t can be approximated by

$$U_t := \max\{X_1, X_2, \ldots, X_{[t\mu^{-1}]}\}$$

and both M_t and U_t have the same a.s. asymptotic behaviour. Best possible results on the a.s. limit behaviour of U_t were obtained by Galambos [11, Theorem 4.3.1, Corollary 4.3.1]. For the sake of completeness we translate his results to M_t:

Theorem 1. *Let $1-F(x)=x^{-\alpha}/L(x)$ and suppose that $\mu<\infty$.*

Let g be a positive increasing function with $g(t)\uparrow\infty$ as $t\to\infty$, and define

$$I(g)=\int_1^\infty \frac{\mathrm{d}t}{g(t)} \qquad J(g)=\int_1^\infty \exp\left(\frac{x}{g(x)}\right)\frac{\mathrm{d}x}{g(x)}.$$

Then

$$P\{M_t \geq (g(t/\mu))^{1/\alpha}L_\alpha(g(t/\mu)) \text{ i.o.}\} = \begin{cases} 1 & \text{if } I(g)=\infty, \\ 0 & \text{if } I(g)<\infty. \end{cases}$$

If furthermore $t/g(t)$ is nondecreasing as $t\to\infty$, then

$$P\{M_t \leq (g(t/\mu))^{1/\alpha}L_\alpha(g(t/\mu)) \text{ i.o.}\} = \begin{cases} 1 & \text{if } J(g)=\infty, \\ 0 & \text{if } J(g)<\infty. \end{cases} \tag{2.1}$$

Remarks. 1. Notice that Theorem 1 holds not only for d.f. F with a regularly varying tail, but also extends to d.f. with a more general tail behaviour; see [11].

2. It follows from (2.1) that (cf. [16, p. 212])

$$\liminf_{t\to\infty} \frac{M_t}{L_\alpha\left(\dfrac{t}{\log\log t}\right)} \left(\frac{\log\log t}{t}\right)^{1/\alpha} = \mu^{-1/\alpha} \quad \text{a.s.} \tag{2.2}$$

We now concentrate on the case $\mu = \infty$.

Theorem 2. *Let*

$$1 - F(x) = \frac{x^{-\alpha}}{L(x)}, \quad 0 \leq \alpha \leq 1$$

and suppose that $\mu = \infty$.

A. *If* $0 \leq \alpha < 1$,

$$\limsup_{t\to\infty} \frac{M_t}{t} = 1 \quad a.s.$$

B. (i) *If* $\alpha = 1$,

$$\liminf_{t\to\infty} \frac{M_t}{L_1^{(1)}\left(\dfrac{t}{\log\log t}\right)} \frac{\log\log t}{t} = 1 \quad a.s.$$

with

$$L^{(1)}(x) = \left(L(x) \int_0^x \frac{1}{uL(u)} \,\mathrm{d}u\right)^{-1}.$$

(ii) *If* $0 < \alpha < 1$,

$$\liminf_{t\to\infty} M_t \frac{\log\log t}{t} = \beta(\alpha) \quad a.s.$$

with $\beta(\alpha)$ *the root of the equation*

$$\sum_{k=1}^{\infty} \frac{\beta^k}{k!} \frac{\alpha}{k-\alpha} = 1.$$

(iii) *If* $\alpha = 0$, $1 - F \in \pi R$,

$$\frac{1}{x} \int_0^x u \,\mathrm{d}F(u) = \mathrm{O}\left(\frac{1-F(x)}{\log\log\log x}\right),$$

$$x(1-F(x)) \int_0^x u\left(1 - \log\frac{x}{u}\right) \mathrm{d}F(u) = \mathrm{O}\left(\left(\int_0^x u \,\mathrm{d}F(u)\right)^2\right) \quad (x \to \infty), \tag{2.3}$$

then

$$\liminf_{t\to\infty} \frac{M_t}{a_t} = 1 \quad a.s.$$

where $a_t^{(1)} \le a_t \le a_t^{(2)}$, $(t-a_t^{(1)})Q(a_t^{(1)}) \sim tQ(a_t^{(2)}) \sim \log\log t$,

$$Q(a) = \frac{1}{a} \frac{1-F(a)}{\frac{1}{a}\int_0^a u \,\mathrm{d}F(u)}.$$

Theorem 2.A shows that if $\mu=\infty$ (more precisely if $0\le\alpha<1$), nearly the whole interval $[0, t]$ may be covered by only one renewal period. However we expect this result not to be true in the case $\alpha=1$. A comparison of (2.2) and Theorem 2.B shows that the lower bound for M_t has a completely different behaviour depending on whether $\mu<\infty$ or not. Indeed, in the first case the parameter dependence can be found in the asymptotic constant while in the second case, the parameter appears in the rate. Also note that in case $0<\alpha<1$ the slowly varying function L has no influence.

The case $\alpha=0$ is very special and we have to concentrate on a special subclass of slowly varying functions. The technical condition in (2.3) is easily checked and is satisfied for a large class of functions $1-F\in\pi R$. We remark that the bounds of the rate a_t are only given implicitly in (2.3) since they can vary very much depending on the behaviour of $1-F$, as will be shown by the following examples:

(i) If

$$F(x) = \begin{cases} 0, & x<\mathrm{e}, \\ 1-\exp-\dfrac{\log x}{\log\log x}, & x\ge \mathrm{e}, \end{cases}$$

then

$$\liminf_{t\to\infty} M_t\left(\frac{\log\log t}{t\log\log\log t}\right) = 1 \quad \text{a.s.}$$

(ii) If

$$F(x) = \begin{cases} 0, & x<1 \\ 1-\exp-(\log x)^{\gamma}, & x\ge 1,\ 0<\gamma<1, \end{cases}$$

then

$$\frac{1-\gamma}{2-\gamma} \le \liminf_{t\to\infty} t^{-1}M_t \le \min\{\tfrac{1}{2}, 1-\gamma\} \quad \text{a.s.}$$

(iii) If

$$F(x) = \begin{cases} 0, & x<\mathrm{e}, \\ 1-(\log x)^{-\gamma}, & x\ge\mathrm{e},\ \gamma>0, \end{cases} \tag{2.4}$$

then

$$\liminf_{t\to\infty} t^{-1}M_t = \tfrac{1}{2} \quad \text{a.s.}$$

This last example shows that in case of (2.4), M_t is asymptotically equal to at least half of the whole range t. In order to estimate the remaining part $t - M_t$, we prove the following.

Lemma 3. *Let $M_t^{(1)} \geq M_t^{(2)} \geq \cdots$ be the order statistics of the sequence $X_1, X_2, \ldots, X_{\tau_t}$, $t - S_{\tau_t}$ and let F be given by* (2.4). *Then*

$$\lim_{t\to\infty} \frac{M_t^{(1)} + M_t^{(2)}}{t} = 1 \quad \text{a.s.}$$

Remarks. 1. It is possible to apply Theorems 1 and 2 in order to determine the asymptotic behaviour of

$$\left(\max_{1\leq i\leq n} X_i\right) S_n^{-1} = M(S_n)\cdot S_n^{-1}.$$

This ratio was investigated among others by Darling [7], Arov and Bobrov [1], and Maller and Resnick [13].

It was shown in [13] that

$$M(S_m)\cdot S_m^{-1} \to 0 \text{ a.s.} \quad \text{if } \mu < \infty \quad (\alpha \geq 1)$$

and

$$M(S_m)\cdot S_m^{-1} \to 1 \text{ a.s.} \quad \text{if } \alpha = 0 \text{ and } \int_0^\infty \frac{\int_0^\infty u \,\mathrm{d}F(u)}{x(1-F(x))^2}\,\mathrm{d}F(x) < \infty.$$

Using Theorem 2.B(iii) we can also deal with the intermediate case $0 < \alpha < 1$: let $S_n \leq t < S_{n+1}$, then

$$t^{-1}M(t) = \begin{cases} t^{-1}M(S_n) & \text{if } S_n \leq t \leq \min\{S_{n+1}, S_n + M(S_n)\}, \\ t^{-1}(t - S_n) & \text{if } \min\{S_{n+1}, S_n + M(S_n)\} \leq t \leq S_{n+1}. \end{cases}$$

Consequently,

$$t^{-1}M(t) \geq (S_n + M(S_n))^{-1}M(S_n) = \frac{S_n^{-1}M(S_n)}{1 + S_n^{-1}M(S_n)} \geq \frac{M(S_n)}{2S_n}. \tag{2.5}$$

Since also

$$\liminf_{t\to\infty} \frac{M(t)}{t} \leq \liminf_{n\to\infty} \frac{M(S_n)}{S_n},$$

we have under the conditions of Theorem 2.B.(ii) that

$$\liminf_{n\to\infty} \frac{M(S_n)}{S_n} \log\log S_n = \beta(\alpha) \quad \text{a.s.}$$

2. Several authors (see for example Logan, Mallows, Rice and Shepp [12], McLeish and O'Brien [14] and Csörgő and Horváth [6]) investigated the properties of the sequence

$$S_n^{-1}\left(\sum_{j=1}^{n}|X_j|^P\right)^{1/p}$$

The case $p=\infty$ corresponds to the case discussed above. It is not clear whether our results can be extended to the case $p<\infty$.

3. The above two remarks suggest that the limit behaviour of $t^{-1}M(t)$ and that of $S_n^{-1}M(S_n)$ agree. However it is not the case. As an example we present the following simple result (without proof): Let F be given by (2.4) then

$$\lim_{n\to\infty} S_n^{-1}M(S_n)=1 \quad \text{a.s.}$$

3. Proofs

The proofs of the theorems heavily depend on the following lemmas which are interesting in their own right.

We denote in the sequel $p(t, a)$ as

$$p(t,a)=\begin{cases}P(M_t\leqslant a) & \text{if } t>a,\\ 1 & \text{if } t\leqslant a.\end{cases} \tag{3.1}$$

Lemma 4. *Let $p(t, a)$ be defined in* (3.1). *Then*

$$p(t,a)=\int_0^a p(t-u,a)\,\mathrm{d}F(u).$$

Proof

$$\begin{aligned}P(M_t\leqslant a)&=E\{P(M_t\leqslant a\,|\,X_1)\}\\ &=E\{P(M_{t-X_1}<a, X_1<a\,|\,X_1)\}\\ &=\textstyle\int_0^a p(t-u,a)\,\mathrm{d}F(u) \quad \text{if } 0<a\leqslant t.\end{aligned}$$

This proves the lemma.

Lemma 5. *Let $a>0$ and let $y(a)$ be the root of the equation*

$$\int_0^a (\exp uy(a))\,\mathrm{d}F(u)=1. \tag{3.2}$$

Then

(i) *if* $0<\alpha<1$,

$$y(a)\sim\frac{\beta(\alpha)}{a}\quad(a\to\infty)$$

with $\beta(\alpha)$ *the root of the equation*

$$\sum_{k=1}^{\infty}\frac{\beta^k}{k!}\frac{\alpha}{k-\alpha}=1;$$

(ii) *if* $1\leqslant\alpha<\infty$,

$$y(a)\sim\frac{1-F(a)}{\int_0^a u\,\mathrm{d}F(u)}\quad(a\to\infty);$$

(iii) *if* $\alpha=0$ *and the conditions of Theorem* 2.B (iii) *are satisfied, then*

$$y(a)\sim\frac{1}{a}\log\frac{1-F(a)}{\dfrac{1}{a}\displaystyle\int_0^a u\,\mathrm{d}F(u)}\quad(a\to\infty).\tag{3.3}$$

Proof. (i) Take $a>0$ such that $y(a)\leqslant\beta/a$; then from the inequality

$$0\leqslant u\left(\frac{\beta}{a}-y(a)\right)\leqslant\exp\frac{u\beta}{a}-\exp uy(a),\quad 0\leqslant u\leqslant a,$$

we have that

$$\left(\frac{\beta}{a}-y(a)\right)\cdot\int_0^a u\,\mathrm{d}F(u)\leqslant\int_0^a\left(\exp\frac{u\beta}{a}-\exp uy(a)\right)\mathrm{d}F(u)=-(1-F(a))\exp\beta$$
$$+\frac{\beta}{a}\int_0^a(1-F(u))\exp\frac{u\beta}{a}\,\mathrm{d}u,\tag{3.4}$$

where the last equality resulted from an integration by parts and (3.2). It follows from

$$\sum_{k=1}^{\infty}\frac{\beta^k}{k!}\frac{\alpha}{k-\alpha}=1$$

that β is the unique solution of the integral equation

$$\frac{\exp\beta}{\beta}=\int_0^1 v^{-\alpha}\exp\beta v\,\mathrm{d}v.$$

Using this, we get from (3.4) that

$$\left(\frac{\beta}{a}-y(a)\right)\int_0^a u\,\mathrm{d}F(u)\leqslant\beta(1-F(a))\int_0^1\left(\frac{1-F(av)}{1-F(a)}-v^{-\alpha}\right)\exp\beta v\,\mathrm{d}v.$$

With the regular variation of $1-F$, we have that (see [9])

$$1-\frac{ay(a)}{\beta}\leqslant\frac{a(1-F(a))}{\int_0^a u\,\mathrm{d}F(u)}\cdot\mathrm{o}(1)\quad(a\to\infty)$$
$$=\frac{1-\alpha}{\alpha}\cdot\mathrm{o}(1)\quad(a\to\infty).$$

Hence it follows that $\lim ay(a)=\beta$ if the limit is taken over all points a for which $y(a)\leqslant\beta/a$.

In the same way as above we can show that $\lim ay(a)=\beta$ if the limit is taken over all points a for which $y(a)\geqslant\beta/a$, and part (i) follows.

(ii) Since

$$e^{uy}-1-uy=\int_0^{uy}\int_0^{v} e^{z}\,dz\,dv\begin{cases}\leqslant\frac{1}{2}e^{uy}(uy)^2,\\ \geqslant\frac{1}{2}(uy)^2,\end{cases}$$

it follows from (3.2) that

$$1=F(a)+y\int_0^a u\,dF(u)+R \tag{3.5}$$

with

$$\frac{y^2}{2}\int_0^a u^2\,dF(u)\leqslant R\leqslant\frac{y^2}{2}\int_0^a u^2\,e^{uy}\,dF(u).$$

Moreover, (3.5) implies that

$$y(a)\leqslant f(a):=\frac{1-F(a)}{\int_0^a u\,dF(u)},$$

whence

$$0\leqslant 1-\frac{y(a)}{f(a)}\leqslant\frac{1}{2(1-F(a))}f^2(a)\,e^{af(a)}\int_0^a u^2\,dF(u)=V(a).$$

If $\alpha>1$,

$$\begin{aligned}V(a)&\sim\frac{1}{2\mu^2}(1-F(a))\int_0^a u^2\,dF(u)\\&=\frac{1}{2\mu^2}(1-F(a))(-a^2(1-F(a))+2\int_0^a u(1-F(u))\,du)\\&=\frac{1}{2\mu^2}(-(a(1-F(a)))^2+2a(1-F(a))\int_0^a (1-F(u))\,du)\\&\to 0\quad(a\to\infty).\end{aligned}$$

If $\alpha=1$,

$$\int_0^a u^2\,dF(u)\sim\tfrac{1}{2}a^2(1-F(a))\quad(a\to\infty),$$

whence

$$0\leqslant 1-\frac{y(a)}{f(a)}\leqslant\left(\frac{a(1-F(a))^2}{\int_0^a u\,dF(u)}\right).$$

Since

$$\frac{a(1-F(a))}{\int_0^a u\,\mathrm{d}F(u)}=\left\{\int_0^1 \frac{1-F(av)}{1-F(a)}\,\mathrm{d}v-1\right\}^{-1}\to 0 \quad (a\to\infty),$$

the proof is completed.

(iii) Define $f(a)$ as the solution of the following equation:

$$\frac{1-F(a)}{1/a\int_0^a u\,\mathrm{d}F(u)}=\sum_{k=1}^{\infty}\frac{(af(a))^k}{k!k}. \tag{3.6}$$

Since $1-F\in\pi V$, the left hand side in (3.6) tends to ∞, whence also $af(a)\to\infty$ if $a\to\infty$. Furthermore,

$$\sum_{k=1}^{\infty}\frac{(af(a))^k}{k!k}=\frac{e^{af(a)}}{af(a)}\left(1+\frac{1}{af(a)}+\mathrm{O}\left(\frac{1}{af(a)}\right)^2\right) \quad (a\to\infty), \tag{3.7}$$

so that, with (3.6),

$$\frac{1-F(a)}{1/a\int_0^a u\,\mathrm{d}F(u)}=\frac{\mathrm{e}^{af(a)}}{af(a)}\left(1+\frac{1}{af(a)}+\mathrm{O}\left(\frac{1}{af(a)}\right)^2\right) \quad (a\to\infty).$$

We now take $z(a)$ such that

$$\frac{1-F(a)}{1/a\int_0^a u\,\mathrm{d}F(u)}=\frac{\mathrm{e}^{az(a)}}{az(a)}, \quad a>0.$$

Then for a sufficiently large we have that $f(a)\leqslant z(a)$ and by use of the inequality (still for a large enough)

$$\begin{aligned}(az(a)-af(a))\frac{\mathrm{e}^{af(a)}}{af(a)}&\leqslant\frac{\mathrm{e}^{az(a)}}{az(a)}-\frac{\mathrm{e}^{af(a)}}{af(a)}\\ &=\mathrm{O}\left(\frac{\mathrm{e}^{af(a)}}{(af(a))^2}\right) \quad (a\to\infty),\end{aligned} \tag{3.8}$$

we get that

$$z(a)\sim f(a) \quad (a\to\infty). \tag{3.9}$$

By taking

$$h(a)=\frac{1}{a}\log\left(\frac{1-F(a)}{1/a\int_0^a u\,\mathrm{d}F(u)}\cdot\log\left(\frac{1-F(a)}{1/a\int_0^a u\,\mathrm{d}F(u)}\right)\right), \quad a>0,$$

we have that

$$\frac{\mathrm{e}^{ah(a)}}{ah(a)}=\frac{1-F(a)}{1/a\int_0^a u\,\mathrm{d}F(u)}\left(1+\frac{\log\log\dfrac{1-F(a)}{1/a\int_0^a u\,\mathrm{d}F(u)}}{\log\dfrac{1-F(a)}{1/a\int_0^a u\mathrm{d}F(u)}}\right)^{-1},$$

whence $h(a)\leqslant z(a)$.

Applying the same inequality as in (3.8) then yields that $z(a) \sim h(a)$, which, combined with (3.9), implies that

$$f(a) \sim \frac{1}{a} \log \frac{1-F(a)}{1/a \int_0^a u \, \mathrm{d}F(u)} \quad (a \to \infty).$$

We now show in the second part of the proof that

$$y(a) \sim f(a) \quad (a \to \infty).$$

From integration by parts, we have that

$$\begin{aligned} 1 - &\int_0^a (\exp uf(a)) \, \mathrm{d}F(u) \\ &= (1-F(a)) - af(a) \int_0^1 (F(a) - F(av)) \exp vaf(a) \, \mathrm{d}v \\ &= (1-F(a)) - \left(\int_0^a u \, \mathrm{d}F(u)\right) f(a) \int_0^1 \left(\log \frac{1}{v}\right) (\exp vaf(a)) \, \mathrm{d}v \\ &\quad - af(a) \int_0^1 \left[F(a) - F(av) - \left(\log \frac{1}{v}\right)\left(\frac{1}{a} \int u \, \mathrm{d}F(u)\right)\right] \exp vaf(a)) \, \mathrm{d}v. \end{aligned}$$

Since

$$af(a) \int_0^1 \left(\log \frac{1}{v}\right) (\exp vaf(a)) \, \mathrm{d}v = \sum_{k=1}^{\infty} \frac{(af(a))^k}{k!k}, \tag{3.10}$$

it follows from the choice of $f(a)$ (see (3.6)) and (3.10) that

$$\begin{aligned} 1 - &\int_0^a (\exp uf(a)) \, \mathrm{d}F(u) \\ &= -af(a) \int_0^1 \left[F(a) - F(av) - \left(\log \frac{1}{v}\right)\left(\frac{1}{a} \int_0^a u \, \mathrm{d}F(u)\right)\right] \\ &\qquad (\exp vaf(a)) \, \mathrm{d}v. \end{aligned} \tag{3.11}$$

Now

$$\begin{aligned} F(a) - &F(av) - \left(\log \frac{1}{v}\right)\left(\frac{1}{a} \int_0^a u \, \mathrm{d}F(u)\right) \\ &= C(av) - C(a) + \int_1^v (C(au) - C(a)) \frac{\mathrm{d}u}{u} \end{aligned} \tag{3.12}$$

where $C(a) := 1/a \int_0^a u \, \mathrm{d}F(u)$.

Since $1-F\in\pi R$ we have that $C\in\pi V$ (see (1.9)). It then follows from [2, I, Corollary 3.6] that for every $0<\varepsilon<1$ there exists a constant K such that for a large enough,

$$|C(av)-C(a)|\leq Kv^{-\varepsilon}\cdot\left|\frac{1}{a}\int_0^a u\,\mathrm{d}C(u)\right|$$

uniformly in $v\in(0,1)$.

Using this in (3.12), we get from (3.11) that for a large enough,

$$\begin{aligned}&\left|1-\int_0^a(\exp uf(a))\,\mathrm{d}F(u)\right|\\&\quad\leq K\left|f(a)\cdot\int_0^a u\,\mathrm{d}C(u)\right|\left\{\int_0^1 v^{-\varepsilon}(\exp vaf(a))\,\mathrm{d}v\right.\\&\qquad\qquad\left.+\frac{1}{af(a)}\int_0^1 v^{-1-\varepsilon}(\exp vaf(a)-1)\,\mathrm{d}v\right\}\\&\quad\leq\frac{2K}{1-\varepsilon}\left|\frac{1}{a}\int_0^a u\,\mathrm{d}C(u)\right|\cdot(\exp af(a)).\end{aligned}\tag{3.13}$$

Now take $a>0$ such that $y(a)\geq f(a)$. Then from the inequality

$$u(y(a)-f(a))\,\mathrm{e}^{uf(a)}\leq\mathrm{e}^{uy(a)}-\mathrm{e}^{uf(a)},\quad 0\leq u\leq a,$$

we have that

$$\begin{aligned}\left(\frac{y(a)}{f(a)}-1\right)\cdot f(a)\cdot\int_0^a u\,\mathrm{e}^{uf(a)}\,\mathrm{d}F(u)&\leq\int_0^a(\mathrm{e}^{uy(a)}-\mathrm{e}^{uf(a)})\,\mathrm{d}F(u)\\&=1-\int_0^a\mathrm{e}^{uf(a)}\,\mathrm{d}F(u).\end{aligned}\tag{3.14}$$

Using integration by parts, we get that

$$\int_0^a u\,\mathrm{e}^{uf(a)}\,\mathrm{d}F(u)=a\int_0^1(F(a)-F(av))\,\mathrm{e}^{vaf(a)}(1+vaf(a))\,\mathrm{d}v$$

and with a similar method as above, we can estimate this integral to find that

$$\int_0^a u\,\mathrm{e}^{uf(a)}\,\mathrm{d}F(u)\sim\left(\int_0^a u\,\mathrm{d}F(u)\right)\frac{\mathrm{e}^{af(a)}}{(af(a))}\quad(a\to\infty).$$

Combining this estimate with (3.13) and (3.14), it follows from (2.3) that

$$\lim\frac{y(a)}{f(a)}=1\tag{3.15}$$

if the limit is taken over all points a for which $y(a) \geqslant f(a)$. Take now $a > 0$ such that $y(a) \leqslant f(a)$. Since

$$a(f(a)-y(a))\left(\frac{1}{a}\int_0^a u\, \mathrm{d}F(u)\right) \leqslant \int_0^a \mathrm{e}^{uf(a)}\, \mathrm{d}F(u)-1,$$

it follows from (2.3), (3.13) and the choice of $f(a)$ that

$$\limsup_{a\to\infty} a(f(a)-y(a)) < \infty. \tag{3.16}$$

We now apply the same inequality as in (3.14) with $f(a)$ and $y(a)$ interchanged, then by (3.13) and (3.16),

$$\lim \frac{y(a)}{f(a)} = 1$$

if the limit is taken over all points a for which $y(a) \leqslant f(a)$. Combining this with (3.15) proves the lemma.

Lemma 6. *Let $p(t, a)$ and $y(a)$ be defined as in* (3.1) *and* (3.2). *Then for every $\alpha \geqslant 0$, there exist absolute constants $C_1(\alpha)$, $C_2(\alpha) > 0$ such that*

$$\begin{aligned} &p(t, a) = C_\alpha(t, \alpha)\exp(-ty(a)) && \text{if } \alpha > 0,\\ &C_1(0)\exp(-ty(a)) \leqslant p(t, a) \leqslant C_2(0)\exp(-(t-a)y(a)) && \text{if } \alpha = 0, \end{aligned} \tag{3.17}$$

and

$$C_1(\alpha) \leqslant C_\alpha(t, a) \leqslant C_2(\alpha).$$

Proof. We first consider the case $\alpha > 0$. In this case Lemma 5 implies the existence of two constants $L_1(\alpha) \geqslant 0$ and $L_2(\alpha) > 0$ such that

$$L_1(\alpha) \leqslant ay(a) \leqslant L_2(\alpha).$$

Since, for $t \leqslant a$,

$$\begin{aligned} 1 &= p(t, a)\\ &= C_\alpha(t, a)\exp - ty(a)\begin{cases} \geqslant C_\alpha(t, a)\ \exp - L_2(\alpha),\\ \leqslant C_\alpha(t, a), \end{cases} \end{aligned}$$

it follows that

$$1 \leqslant C(t, a) \leqslant \exp L_2(\alpha). \tag{3.18}$$

We now show that (3.18) also holds for $t > a$. Suppose that this is not the case, then there exists a smallest $t = t_0 > a$ for which $C(t_0, a) = \mathrm{e}^{L_2(\alpha)}$ (or $C(t_0, a) = 1$). By

Lemma 4,

$$\begin{aligned}
p(t_0, a) &= C(t_0, a)\exp(-t_0 y(a)) \\
&= \int_0^a C(t_0-u, a)\cdot(\exp-(t_0-u)y(a))\,\mathrm{d}F(u) \\
&= \exp(-t_0 y(a)) \int_0^a C(t_0-u, a)(\exp uy(a))\,\mathrm{d}F(u) \\
&< \mathrm{e}^{L_2(\alpha)}\exp(-t_0 y(a)) \int_0^a (\exp uy(a))\,\mathrm{d}F(u) \\
&= \mathrm{e}^{L_2(\alpha)}\exp(-t_0 y(a)),
\end{aligned}$$

whence

$$C(t_0, a) < \mathrm{e}^{L_2(\alpha)},$$

giving a contradiction. Hence (3.18) holds for all t and a.

The proof of the case $\alpha = 0$ is completely similar. The details are left to the reader.

Proof of Theorem 2.A. Let

$$n_i(i\colon 1, 2, \ldots) = \begin{cases} (2^i)^{1/\alpha} L_\alpha(2^i) & \text{if } 0<\alpha<1, \\ L^{\mathrm{inv}}(2^i) & \text{if } \alpha = 0, \end{cases}$$

where L^{inv} is the inverse of L. Notice that L^{inv} is well defined, since if $\alpha = 0$, L is monotone.

Further, let

$$n_i^* := \frac{1}{1-F(n_i)}, \quad i = 1, 2, \ldots.$$

Define the events $(i = 1, 2, \ldots)$

$$A_i = A_i(\varepsilon) = \left\{ \max_{1\le j\le n_i^*} X_j \ge n_i,\ \sum_{j=1}^{n_i^*} X_j - \max_{1\le j\le n_i^*} X_i \le \varepsilon n_i \right\},$$

$$B_i = \left\{ \max_{n_i^*\le j<n_{i+1}^*} X_j \ge n_{i+1},\ \sum_{j=n_i^*}^{n_{i+1}^*-1} X_j - \max_{n_i^*\le j<n_{i+1}^*} X_j \le \varepsilon n_{i+1} \right\}$$

and

$$B_i^{(k)} = \left\{ X_k \ge n_{i+1},\ \sum_{\substack{j=n_i^*\\ j\ne k}}^{n_{i+1}^*-1} X_j \le \varepsilon n_{i+1} \right\},\ n_i^* \le k < n_{i+1}^*.$$

If $0<\alpha<1$, it follows from Feller [10] that

$$P(B_i^{(k)}) \sim \frac{\Lambda_\alpha(\varepsilon)}{n_{i+1}^*} \quad (i\to\infty) \tag{3.19}$$

where Λ_α is a stable law of index α.

If $\alpha=0$, we have from Darling [7] that

$$P(B_i^{(k)}) \sim \frac{e^{-1/2}}{n_{i+1}^*} \quad (i\to\infty). \tag{3.20}$$

Since the events $B_i^{(k)}$ $(k=n_i^*,\dots,n_{i+1}^*-1)$ are disjoint (provided that $0<\varepsilon<1$), and since

$$\sum_{k=n_i^*}^{n_{i+1}^*-1} B_i^{(k)} = B_i,$$

it follows from (3.19) and (3.20) that

$$P(B_i)\to C>0 \quad \text{if } 0\leq\alpha<1.$$

Since furthermore the events $(B_i)_{i=1}^\infty$ are mutually independent, the Borel-Cantelli Lemma implies that the events B_i occur infinitely often (for any $0<\varepsilon<1$), which in turn implies that the events A_i occur infinitely often.

Let $N_1=N_1(\omega)<N_2=N_2(\omega)<\cdots$ be a random sequence of integers such that A_{N_i} occurs for $i=1,2,\dots$. Define $t_i=S_{N_i^*}$ where $N_i^*=(1-F(N_i))^{-1}$.

Then

$$t_i\leq \varepsilon N_i + \max_{1\leq j\leq N_i^*} X_j$$

and

$$M_{t_i} = \max_{1\leq j\leq N_i^*} X_j.$$

Consequently $t_i^{-1}M_{t_i}\geq(1+\varepsilon)^{-1}$ for any $1>\varepsilon>0$. Since $t^{-1}M_t\leq 1 (t>0)$, the proof is finished.

Proof of Theorem 2.B. Take $\varepsilon>0$. Choose $a_\alpha(t)$ $(b_\alpha(t))$ as the solution of

$$y(a_\alpha(t))=(1+\varepsilon)\frac{\log\log t}{t} \quad \left(y(b_\alpha(t))=(1-\varepsilon)\frac{\log\log t}{t}\right)$$

if $\alpha>0$ and let $a_0(t)(b_0(t))$ be defined by

$$(t-a_0(t))y(a_0(t))=(1+\varepsilon)\log\log t,$$

$$ty(b_0(t))=(1-\varepsilon)\log\log t.$$

Now let f be defined as

$$f(a)=\begin{cases} \dfrac{1}{a}\log\dfrac{1-F(a)}{\dfrac{1}{a}\displaystyle\int_0^a u\,\mathrm{d}F(u)} & \text{if } \alpha=0, \quad (3.21)\\[2ex] \dfrac{\beta(\alpha)}{a} & \text{if } 0<\alpha<1, \quad (3.22)\\[2ex] \dfrac{1-F(a)}{\displaystyle\int_0^a u\,\mathrm{d}F(u)} & \text{if } \alpha=1, \quad (3.23)\end{cases}$$

and take $c_\alpha(t)$ such that

$$(1+\varepsilon)^{-1}y(a_\alpha(t))=f(c_\alpha(t))=(1-\varepsilon)^{-1}y(b_\alpha(t)) \quad \text{for } 0\leqslant\alpha\leqslant 1.$$

It then follows from Lemma 5 that

$$\frac{f(a_\alpha(t))}{f(c_\alpha(t))}=\frac{f(a_\alpha(t))}{y(a_\alpha(t))}(1+\varepsilon)\to(1+\varepsilon) \quad (t\to\infty).$$

Since f in (3.21)-(3.23) is regularly varying with a non-zero index, it follows from [9, Corollary 1.2.1] that

$$\lim_{t\to\infty}\frac{a_\alpha(t)}{c_\alpha(t)}=\begin{cases}(1+\varepsilon)^{-1} & \text{if } 0\leqslant\alpha<1,\\ (1+\varepsilon)^{-\alpha} & \text{if } \alpha=1.\end{cases} \tag{3.24}$$

In the same way we can show that

$$\lim_{t\to\infty}\frac{b_\alpha(t)}{c_\alpha(t)}=\begin{cases}(1-\varepsilon)^{-1} & \text{if } 0\leqslant\alpha<1\\ (1-\varepsilon)^{-\alpha} & \text{if } \alpha=1.\end{cases} \tag{3.25}$$

By Lemma 6 and the choice of $a_\alpha(t)$, we have that, for $0\leqslant\alpha\leqslant 1$,

$$p(t, a_\alpha(t))\leqslant C_2(\alpha)(\log t)^{-(1+\varepsilon)}.$$

Now take

$$t_k=\theta^k, \quad \theta>1, \ k=1,2,\ldots;$$

then

$$P(M_{t_k}<a_\alpha(t_k))\leqslant C_2(\alpha)(k\log\theta)^{-(1+\varepsilon)}$$

whence by the Borel-Cantelli Lemma,

$$M_{t_k}\geqslant a_\alpha(t_k) \quad \text{a.s.}$$

for all but finitely many k.

Since M_t is a decreasing function of t, for any $t_k \leqslant t < t_{k+1}$, we obtain that $M_t \geqslant a_\alpha(t_k)$. Choosing θ sufficiently close to 1, we get that

$$M_t \geqslant (1-\varepsilon)a_\alpha(t) \quad \text{a.s.}$$

if t is big enough.

Now using (3.24) and letting $\varepsilon \to 0$ gives

$$\liminf_{t\to\infty} \frac{M_t}{c_\alpha(t)} \geqslant 1 \quad \text{a.s.} \tag{3.26}$$

We now turn to the proof of the converse in (3.26) and we show that

$$\liminf_{t\to\infty} \frac{M_t}{c_\alpha(t)} \leqslant 1 \quad \text{a.s.} \tag{3.27}$$

Let $\alpha > 0$, $u_k = \exp(k \log k)$ and let m_k be the smallest integer for which

$$S_{m_k} \geqslant u_k.$$

Observe that

$$u_k = o(b_\alpha(u_{k+1})) \quad (k \to \infty)$$

and

$$\begin{aligned} P(S_{m_k} \geqslant \varepsilon b_\alpha(u_{k+1})) &\leqslant P(X_{m_k} \geqslant \varepsilon b_\alpha(u_{k+1}) - u_k) \\ &\leqslant P(X_{m_k} \geqslant \frac{\varepsilon}{2} b_\alpha(u_{k+1})) \\ &\leqslant P\left(X_1 \geqslant \frac{\varepsilon}{2} b_\alpha(u_{k+1}) \,|\, X_1 \geqslant u_k\right) \\ &\leqslant P(X_1 \geqslant \frac{\varepsilon}{2} \frac{b_\alpha(u_{k+1}))}{P(X_1 \geqslant u_k)}. \end{aligned}$$

Using (3.25) and the definition of $c_\alpha(t)$, it follows with an application of the Borel-Cantelli Lemma that for every $\varepsilon > 0$,

$$S_{m_k} \leqslant \varepsilon \cdot b_\alpha(u_{k+1}) \quad \text{a.s.} \tag{3.28}$$

for all but finitely many k.

Let

$$M(S_{m_k}, u_{k+1}) = \max\{X_{m_{k+1}}, X_{m_{k+2}}, \ldots, X_{\tau_{u_{k+1}}}, u_{k+1} - S_{\tau_{u_{k+1}}}\},$$

then by Lemma 6 and the choice of $b_\alpha(t)$,

$$P(M(S_{m_k}, u_{k+1}) < b_\alpha(u_{k+1})) \geqslant c(\alpha)(\log u_{k+1})^{-(1-\varepsilon)}$$

where $c(\alpha)$ is some absolute constant.

Hence by the Borel-Cantelli Lemma, we have that

$$M(S_{m_k}, u_{k+1}) \leq b_\alpha(u_{k+1}) \quad \text{i.o. a.s.}$$

which together with (3.28) and (3.25) implies (3.27). In case $\alpha = 0$, let $u_k = L^{\text{inv}}(\exp(k \log k))$ and we get the result similarly.

Proof of Lemma 3. Let

$$T = T(t) = [2(\log t)^\gamma \log\log t];$$

then

$$P(\max(X_1, \ldots, X_T) < 2t) = (F(2t))^T = \left(1 - \frac{1}{(\log 2t)^\gamma}\right)^T \sim \frac{1}{(\log t)^2} (t \to \infty).$$

Now take $t_k = 2^k$; then

$$\max(X_1, \ldots, X_{T(t_k)}) \geq 2t_k \quad \text{a.s.}$$

if k is big enough. Further let $t_k \leq t < t_{k+1}$; then

$$\max(X_1, \ldots, X_{T(t)}) \geq \max(X_1, \ldots, W_{T(t_k)}) \geq 2t_k \geq t.$$

Hence

$$\max(X_1, \ldots, X_{T(t)}) \geq t \quad \text{a.s.}$$

if t is big enough. This in its turn implies that

$$\tau_t \leq T. \tag{3.29}$$

We now show that

$$\xi_t := \#\left\{i\colon i \leq T(t), \alpha(t) = \frac{t}{(\log t)^{2\gamma}} \leq X_i \leq t = \beta(t)\right\} \leq 1 \quad \text{a.s.}$$

if t is big enough. Let

$$\xi_t^* := \#\{i\colon i \leq T(2t), \alpha(t) \leq X_i \leq \beta(2t)\}.$$

Since

$$P_t := P(\alpha(t) \leq X_i \leq \beta(2t)) = \left(\log\left(\frac{t}{(\log t)^{2\gamma}}\right)\right)^{-\gamma} - (\log 2t)^{-\gamma}$$

$$\sim 2\gamma^2 \frac{\log\log t}{(\log t)^{\gamma+1}} \quad (t \to \infty),$$

we have that

$$\begin{aligned} P(\xi_t^* > 1) &= 1 - (1 - P_t)^{T(2t)} - T(2t)(1 - P_t)^{T(2t)-1} \cdot P_t \\ &\sim \frac{(T(2t)P_t)^2}{2} \\ &\sim 8\gamma^4 \frac{(\log\log t)^4}{(\log t)^2} \quad (t \to \infty). \end{aligned}$$

Let $t_k = 2^k$. Then $\xi^*_{t_k} \leq 1$ a.s. if k is big enough. Now take

$$t_k \leq t \leq t_{k+1}.$$

Since

$$T(t) \leq T(2t_k)$$

and

$$\alpha(t_k) \leq \alpha(t) < \beta(t) \leq \beta(2t_k),$$

we obtain that

$$\xi_t \leq \xi^*_{t_k} \leq 1.$$

Since $\tau_t \leq T(t)$ (see (3.29)) we have that $S_{T(t)} \geq t$. Clearly $M_t^{(2)} < t$ and the sum of those elements of the sequence $X_1, X_2, \ldots, X_{\tau_t}, t - S_{\tau_t}$ which are smaller than $\alpha(t)$ is $o(t)$. Hence the fact that $\xi_t \leq 1$ a.s. implies the lemma.

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